Buoyancy Effects on a Turbulent Shear Flow

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The effects of buoyancy on the structure of a turbulent flow with uniform velocity, temperature, and concentration gradients in the vertical direction are studied theoretically on the basis of a statistical turbulent-flow model. The study shows that the three components of the turbulent kinetic energy, the mean square temperature fluctuation, the Reynolds stress, the turbulent heat transfer, and the turbulent mass transfer are all enhanced under unstable conditions and suppressed under stable conditions. The turbulent heat transfer is more strongly affected by the buoyancy than the Reynolds stress, so that the ratio of the eddy conductivity to the eddy viscosity decreases at increasing Richardson numbers. The buoyancy affects the turbulent heat transfer and the turbulent mass transfer in the same manner. Comparison with experimental data shows satisfactory agreement for most of the turbulent properties.

Nomenclature

b = constant defined by Eq. (29)
c = a scalar quantity
D = molecular diffusivity
E = $\frac{1}{3}\langle U^2 + V^2 + W^2 \rangle$ = turbulent kinetic energy
E_v, E_w = parameters in the distribution function for velocity, defined by Eqs. (24) and (26)
F = one-point distribution function for a scalar quantity
f = one-point distribution function for velocity; also function defined by Eq. (37)
$f^{(n)}$ = n – point distribution function
g_i = gravity force in the ith direction
k = reaction rate coefficient
K_i = function defined by Eq. (9)
K_H = eddy conductivity
K_M = eddy viscosity
K_S = eddy diffusivity
L = characteristic length of the flow
m = reaction rate order
n = species concentration
n_1, n_2 = parameters in the distribution function for concentration; defined by Eq. (23)
$\langle Q \rangle$ = $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} QfdUdVdW$ = one-point ensemble average of Q
$\bar{Q}$ = n-point ensemble average of Q; see Eq. (4)
R_f = flux Richardson number
Ri = gradient Richardson number
t = time
U = $u - \langle u \rangle$ = fluctuating velocity vector
U_i = ith component of the fluctuating velocity
u_i = ith component of the instantaneous velocity
u_1, u_2 = parameters in the distribution function for velocity, defined by Eqs. (24) and (26)
U, V, W = fluctuating velocity components in the x,y,z directions
u, v, w = instantaneous velocity components in the x,y,z directions
x,y,z = Cartesian coordinates in the streamwise, vertical, and spanwise directions
α = function defined in Eq. (37)
β = large-scale relaxation rate; defined in Eq. (28)
η = function defined in Eq. (37)
θ = temperature
θ_1, θ_2 = parameters in the distribution function for temperature; defined by Eq. (22)
Λ = scale length of large energy containing eddies
λ = scale length of small dissipative eddies
ν = kinematic viscosity
ϕ = function defined in Eq. (37)
ψ = function defined in Eq. (37)

Superscripts and Subscripts

* = dimensionless double correlation; see Eqs. (46-53)
0 = at zero Richardson number

Introduction

It has long been recognized that the buoyancy force due to density stratification has pronounced effects on the turbulent structure and, consequently, on the transport processes. Turbulent transports of momentum, heat, and chemical species in stratified fields constitute the basis of many current engineering problems, such as those associated with atmospheric and aquatic dispersion. There have been a number of laboratory measurements of the buoyancy effects on the turbulent structure and the turbulent transport coefficients.[1-7] These experimental data invariably show that, for positive values of the Richardson number, all turbulent fluctuating properties are suppressed by the action of the buoyancy force. This reduction of turbulence is the result of the interaction between the buoyancy force and the fluctuation of the vertical component of the velocity. Because of the tendency of the turbulent field toward isotropy, the fluctuation of the vertical component subsequently interacts with the fluctuation of the other two components, causing them to subside as well. These physical considerations suggest that an adequate analysis of the transport properties must include a more detailed treatment of the effect of the interaction between the buoyancy force and the turbulent fluctuation than that which can be done with a conventional mixing-length theory.

The results of many classical statistical and experimental studies are found in the literature,[8-12] which describes various

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fundamental aspects of the stratified atmosphere turbulence. These studies discuss the spectral distribution of two-point correlations as affected by buoyancy and, in particular, the general role of the Richardson number in determining the structure of a turbulence field. These fundamental statistical and experimental studies give much needed understanding of the nature of the turbulent-flow problem. However, as in most turbulent-flow problems of engineering interest, the results of these studies cannot be directly applied to the quantitative analysis of stratified turbulent-shear-flow problems. For instance, in spite of these classical statistical studies, analyses of the turbulent transports of momentum and heat in stratified-shear flowfields of engineering interest have, in the past, almost exclusively depended on empirical approaches, such as those based on the Prandtl's mixing length theory.[11,13-15]

Several turbulent-flow theories[16-22] have been developed in recent years which attempt to incorporate into the engineering shear-flow analyses certain salient findings of classical-statistical studies. Donaldson's work[21] constitutes the first application of one of these recent theories to the study of a stratified shear flow where the second-order one-point moment equation has been included.

The present paper employs the kinetic-theory approach developed by Chung[23-27] in the analysis of a stratified shear flow. The statistical nature of interaction of the buoyancy with turbulent fluctuations and its effects on the turbulence energy and trnasport can be readily accounted for in this approach. The particular problem analyzed here is the shear flow with uniform mean velocity gradient stratified by a uniform mean density (temperature) gradient for which experimental results are available.[1] All pertinent turbulence quantities, such as the streamwise and vertical components of the turbulent energy, as well as the mean transport rates of momentum, heat, and chemical species will be obtained as functions of the Richardson number. These theoretical results will be compared with the existing experimental data.

Theory

Most recent theories of turbulent flows[16-22] are based on the closure of second-order moment equations. In these theories, the various moments (one-point averaged quantities), such as $<U_iU_j>$, $<c^2>$, $<cU_j>$, and $<c^2U_j>$, are considered as the independent functions requiring separate governing equations. Closure is then applied to each of the moment equations separately.

In the present theory, kinetic equations are derived which govern the probability density functions (herein called the distribution functions) of the fluid elements and scalar quantities, such as temperature and chemical species, contained in the fluid elements. Once these distribution functions are determined for a given problem, all one-point averaged quantities (moments) of engineering interest naturally result as

$$<U_iU_j> = \int f(t,x,U)\, U_iU_j dU$$
$$<U_iU_jU_k> = \int f(t,x,U)\, U_iU_jU_k dU \tag{1}$$

$$<cU_j> = \int f(t,x,U)\, c(t,x,U)\, U_j dU$$
$$<c^2> = \int f(t,x,U)\, c^2(t,x,U)\, dU \tag{2}$$

As it is implicit in Eqs. (1) and (2), the probability of finding a scalar quantity is closely related to that of finding a fluid element containing the scalar quantity and, therefore, distribution function of the scalar quantity is expressed in terms of that of the fluid element as

$$F(t,x,c)\,dc \to f(t,x,U)\, c(t,x,U)\, dU \tag{3}$$

Note that, without the above relationship, one must generate kinetic equations for distribution functions of all possible combinations of c's and U's, which would render the analysis intractable.

We shall now briefly summarize the development of the kinetic equations given in Chung.[23-27] We first consider an n-point distribution function of fluid elements

$$f^{(n)}(t,x_1,x_2,\ldots x_n,v_1,v_2,\ldots v_n) \times dv_1 dv_2,\ldots dv_n$$

where n may be as large as is necessary to describe the turbulence field correctly. $f^{(n)}$ denotes, for the entire flowfield at time t, the simultaneous probability of finding the velocity of fluid element at x_1 to be between v_1 and v_1+dv_1, that of fluid element at x_2 to be between v_2 and v_2+dv_2, etc. Then the detailed ensemble average of any function

$$Q[v_1(x_1,t)v_2(x_2,t),\ldots v_n(x_n,t)]$$

is given by

$$\bar{Q} = \int f^{(n)} Q\, dV_1 dV_2 \ldots dV_n \tag{4}$$

The bar over a function is employed to represent the particular ensemble average defined by Eq. (4).

Following the formalism of statistical mechanics, the lower hierarchy distribution functions are defined in terms of $f^{(n)}$ and the Dirac delta function as

$$f^{(1)}(t,x_1,U_1) = \overline{\delta[V_1(x_1,t)-U_1]} \tag{5a}$$

$$f^{(2)}(t,x_1,x_2,U_1,U_2) = \overline{\delta[V_1(x_1,t)-U_1]\,\delta[V_2(x_2,t)-U_2]} \tag{5b}$$

et cetera.

The one-point distribution function $f^{(1)}(t,x_1,U_1)$ defined in Eq. (5a) is the same as the present distribution function $f(t,x,U)$. By putting $Q=c$, we obtain, analogous to Eq. (5a)

$$cf = \overline{c\delta[V_1(x_1,t)-U_1]}$$

Differentiating this relationship with respect to time, we obtain

$$\frac{\partial cf}{\partial t} = \overline{\partial/\partial t[c\delta(V_1-U_1)]}$$
$$= -\overline{c\frac{\partial V_1}{\partial t}\cdot\frac{\partial}{\partial U_1}\delta(V_1-U_1)} + \overline{\frac{\partial c}{\partial t}\delta(V_1-U_1)} \tag{6}$$

Now, we obtain the expressions for $\partial V_1/\partial t$ and $\partial c/\partial t$ from the Navier-Stokes and species-conservation equations, respectively, in terms of the space derivatives. Substitution of these expressions into Eq. (6) and the subsequent manipulation result in

$$\frac{\partial fc}{\partial t} = -U_j\frac{\partial fc}{\partial x_j} + f\dot{\omega} + (4\pi)^{-1}\frac{\partial}{\partial U_j}$$
$$\times\left[c\int\frac{\partial}{\partial x_k}|x-x_2|^{-1}\left(U_{2m}\frac{\partial}{\partial x_{2m}}\right)^2 f^{(2)}\,dx_{2k}\,dU_{2j}\right]$$
$$-\overline{\nu c\frac{\partial^2 v_j}{\partial x_k\partial x_k}\frac{\partial}{\partial U_j}\delta(V-U)} + \overline{D\frac{\partial^2 c}{\partial x_k\partial x_k}\delta(V-U)} \tag{7}$$

where the subscript 1 has been dropped for convenience.

In Eq. (7), $\dot{\omega}$ denotes the mass rate of production of the species by chemical reaction per unit mass of the fluid mixture. Equation (7) degenerates to the equation first derived by Lundgren[19] when the flowfield consists of one chemically inert gas, that is, when $c=1$ and $\dot{\omega}=0$. Equation (7) is exact, and it has been derived by applying the formalism of statistical mechanics to the Navier-Stokes and species conservation equations. However, as with all fundamental equations of statistical mechanics, Eq. (7) is not closed. The closing mainly entails quantifying the term in the square bracket which contains the two-point distribution function $f^{(2)}$. This term in the square bracket represents the eddy scrambling, which degenerates the nonequilibrium anisotropic eddies toward equilibrium (isotropy). When $c=1$, Lundgren[28] simply replaced this term by a Krook model. No attempt was made by Lundgren to justify the replacement except the fact that this term is analogous to the collision integral of the Boltzmann equation. In Chung's theory, this term is approximated by a Fokker-Planck model. It is known[29,30] that when the turbulence Reynolds number is sufficiently large, a statistical separation exists between the nonequilibrium degrees of freedom representing the nonequilibrium energy-containing eddies and the equilibrium degrees of freedom representing the smaller near-equilibrium eddies, which causes the particular energy cascading characteristics, etc. This statistical property of interactions was shown[24] to satisfy the basic criteria of the generalized Brownian stochastics.[31] This enabled the derivation of the Fokker-Planck form for the square-bracketed terms in Eq. (7). Note that the Brownian stochastics are employed in this theory only to approximate the known statistical result on the spectral plane of the interaction of the various degrees of freedom obtained in the classical statistical theories.

With the square-bracketed terms of Eq. (7) approximated by the Fokker-Planck form, and with further manipulation of the equation, the kinetic equation for fc is derived as

$$\frac{\partial cf}{\partial t} + U_j \frac{\partial cf}{\partial x_j} + \frac{\partial}{\partial U_j}(cfK_j) = \beta\left[\frac{\partial}{\partial U_j}(cfU_j) + \frac{<U_k U_k>}{3}\frac{\partial^2 cf}{\partial U_j \partial U_j}\right] + \dot{\omega} f \tag{8}$$

where K_j represents the molecular dissipation and body force.

Now we consider the present stratified flow. For the present problem with no mean pressure gradient but with the buoyancy body force created by a temperature field only, we model K_j as

$$K_j(x,U) = -(\nu/\lambda^2)U_j + g_j[(\theta - <\theta> / <\theta>] \tag{9}$$

The first term in Eq. (9) corresponds to the dissipation due to molecular viscosity with the dissipation scale denoted by λ. For flows at very high Reynolds numbers, the rate of energy dissipation by the small eddies is controlled by the rate of energy supply from the large eddies on the basis of the universal equilibrium theory. Chung[25] suggested that the dissipation term be set equal to the relaxation term of the large eddies, viz., $-\beta U_i$. In anticipation of comparing the present analysis with available experimental data which were taken at moderate Reynolds numbers, universal equilibrium may not be attained, and the form appearing in Eq. (9) was used instead. Finally, the last term in Eq. (9) describes the effect of buoyancy on the turbulent motion through the interaction of the gravity force component g_i with the temperature field.

Combining Eqs. (8) and (9), and by setting $c=1,\theta,$ and n, successively, we obtain the following three kinetic equations respectively governing the distribution functions of the fluid elements, temperature, and chemical species.

$$\frac{\partial f}{\partial t} + U_j\frac{\partial f}{dx_j} = -g_j\frac{\partial}{\partial U_j}\left\{f\frac{(\theta - <\theta>}{<\theta>}\right\} + \beta\left\{\frac{\partial}{\partial U_j}(fU_j) + E\frac{\partial^2 f}{\partial u_k \partial u_k}\right\} + \frac{\nu}{\lambda^2}\frac{\partial}{\partial U_j}(fU_j) \tag{10}$$

$$\frac{\partial(\theta f)}{\partial t} + U_j\frac{\partial(\theta f)}{\partial x_j} = -g_j\frac{\partial}{\partial U_j}\left\{\theta f\frac{(\theta - <\theta>)}{<\theta>}\right\} + \beta\left\{\frac{\partial}{\partial U_j}(\theta f U_j) + E\frac{\partial^2(\theta f)}{\partial U_k \partial U_k}\right\} + \frac{\nu}{\lambda^2}\left\{\frac{\partial}{\partial U_j}(\theta f U_j) - f(\theta - <\theta>)\right\} \tag{11}$$

$$\frac{\partial(nf)}{\partial t} + U_j\frac{\partial(nf)}{\partial x_j} = -g_j\frac{\partial}{\partial U_j}\left\{nf\frac{(\theta - <\theta>)}{<\theta>}\right\} + \beta\left\{\frac{\partial}{\partial U_j}(nfU_j) + E\frac{\partial^2(nf)}{\partial U_k \partial U_k}\right\} + \frac{\nu}{\lambda^2}\left\{\frac{\partial}{\partial U_j}(nfU_j) - f(n - <n>)\right\} + kn^m f \tag{12}$$

Here, $f(x_j,U_j,t)$ is the distribution function for the probability that a fluid element lies between x_j,U_j and x_j+dx_j,U_j+dU_j. The distribution functions for the temperature θf and species concentration nf are similarly defined. Both the molecular Prandtl number and Schmidt number are assumed of order one, and an mth order dissociation reaction is assumed with a reaction rate coefficient k. Heat generation due to chemical reaction is assumed negligible.

Analysis

Moment Equations

Equations (10-12) are in the form of three simultaneous integro-differential equations and in principle can be solved directly. All observable properties of the turbulent reacting flow can then be computed from the three distribution functions. To simplify the mathematics, however, solution by the moment method proposed by Chung[25] was employed. Thus, analogous to solving the Boltzmann equation of the kinetic theory of gases by moment methods[32-34] and instead of solving for the three distribution functions directly, various moments of the distribution functions will be solved. While the turbulent structure will not be determined in minute detail by such a method, the moments in fact represent the various observable properties of the turbulent flow. Their solutions do provide an adequate description of flow properties and facilitate comparison with experimental data.

The generalized moment equations of the Fokker-Planck equations are derived by multiplying individually Eqs. (10-12) by a general function $Q(x_i,U_i,t)$ and integrating over the entire velocity space. One obtains

$$\frac{\partial}{\partial t}\int Qf dU - \int f\frac{\partial Q}{\partial t}dU + \frac{\partial}{\partial x_k}\int(<u_k> + U_k)Qf dU - \int(<u_k> + U_k)f\frac{\partial Q}{\partial x_k}dU + \int\frac{\partial Q}{\partial U_i}\left\{(<u_k> + U_k)\frac{\partial <u_i>}{\partial x_k}\right\}f dU - \int\frac{\partial Q}{\partial U_k}\left[\frac{\theta - <\theta>}{<\theta>}\right]g_k f dU = -\left[\beta + \frac{\nu}{\lambda^2}\right]\int U_k\left[\frac{\partial Q}{\partial U_k}\right]f dU - \beta E\int\frac{\partial Q}{\partial U_k}\frac{\partial f}{\partial U_k}dU \tag{13}$$

$$\frac{\partial}{\partial t}\int Q\theta f\mathrm{d}U-\int \theta f\frac{\partial Q}{\partial t}\mathrm{d}U+\frac{\partial}{\partial x_k}\int (<u_k>+U_k)Q\theta f\mathrm{d}U-\int (<u_k>+U_k)\theta f\frac{\partial Q}{\partial x_k}\mathrm{d}U+\int \frac{\partial Q}{\partial U_i}\left\{(<u_k>+U_k)\frac{\partial <u_i>}{\partial x_k}\right\}\theta f\mathrm{d}U-\int\frac{\partial Q}{\partial U_k}\left\{\frac{\theta-<\theta>}{<\theta>}\right\}g_k\theta f\mathrm{d}U$$

$$=-\left[\beta+\frac{\nu}{\lambda^2}\right]\int U_k\frac{\partial Q}{\partial U_k}\theta f\mathrm{d}U-\beta E\int\frac{\partial Q}{\partial U_k}\frac{\partial(\theta f)}{\partial U_k}\mathrm{d}U-\frac{\nu}{\lambda^2}\int(\theta-<\theta>)Qf\mathrm{d}U \quad (14)$$

$$\frac{\partial}{\partial t}\int Qnf\mathrm{d}U-\int nf\frac{\partial Q}{\partial t}\mathrm{d}U+\frac{\partial}{\partial x_k}\int (<u_k>+U_k)\times Qnf\mathrm{d}U-\int (<u_k>+U_k)nf\frac{\partial Q}{\partial x_k}\mathrm{d}U+\int \frac{\partial Q}{\partial U_i}\left\{(<u_k>+U_k)\frac{\partial <u_i>}{\partial x_k}\right\}nf\mathrm{d}U-\int\frac{\partial Q}{\partial U_k}\left\{\frac{\theta-<\theta>}{<\theta>}\right\}g_k nf\mathrm{d}U$$

$$=-\left[\beta+\frac{\nu}{\lambda^2}\right]\int U_k\frac{\partial Q}{\partial U_k}nf\mathrm{d}U-\beta E\int\frac{\partial Q}{\partial U_k}\frac{\partial(nf)}{\partial U_k}\mathrm{d}U-\frac{\nu}{\lambda^2}\int(n-<n>)Qf\mathrm{d}U+k\int Qn^m f\mathrm{d}U \quad (15)$$

For no buoyancy force, this set of moment equations has been shown by Chung to be essentially equivalent to the moment equations derived from the Navier-Stokes equations.[23,24] As an application of the statistical theory to the turbulent flow with buoyancy force acting and for comparison with available experimental data, the generalized moment equations are specialized to the case of a steady homologous turbulent shear flow with uniform vertical gradients of velocity, temperature, and species concentration in a nonreacting environment. Thus we let

$$g_x=g_z=<v>=<w>=\partial/\partial x=\partial/\partial z=\partial/\partial t=0$$

By setting Q equal to U^2, V^2, W^2 successively in Eq. (13) and equal to V in both Eqs. (14) and (15), the following moment equations result

$$\frac{\mathrm{d}}{\mathrm{d}y}\int VU^2f\mathrm{d}U+2\frac{\mathrm{d}<u>}{\mathrm{d}y}\int VUf\mathrm{d}U=-\frac{2\nu}{\lambda^2}\int U^2f\mathrm{d}U+2\beta\int\left\{{\textstyle\frac{1}{3}}(U^2+V^2+W^2)-U^2\right\}f\mathrm{d}U \quad (16)$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\int V^3f\mathrm{d}U-\frac{2g}{<\theta>}\int V\theta f\mathrm{d}U=-\frac{2\nu}{\lambda^2}\int V^2f\mathrm{d}U+2\beta\int\left\{{\textstyle\frac{1}{3}}(U^2+V^2+W^2)-V^2\right\}f\mathrm{d}U \quad (17)$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\int VW^3f\mathrm{d}U=-\frac{2\nu}{\lambda^2}\int W^2f\mathrm{d}U+2\beta\int\left\{{\textstyle\frac{1}{3}}(U^2+V^2+W^2)-W^2\right\}f\mathrm{d}U \quad (18)$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\int V^2\theta f\mathrm{d}U-\frac{g}{<\theta>}\int(\theta-<\theta>)^2f\mathrm{d}U=-\left[\beta+\frac{2\nu}{\lambda^2}\right]\int V\theta f\mathrm{d}U \quad (19)$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\int V^2nf\mathrm{d}U-\frac{g}{<\theta>}\int(\theta-<\theta>)(n-<n>)f\mathrm{d}U=-\left[\beta+\frac{2\nu}{\lambda^2}\right]\int Vnf\mathrm{d}U \quad (20)$$

Equations (16-20) represent physically the governing equations for the turbulent kinetic energies in the x,y,z directions and for the turbulent heat flux and species flux in the y direction, respectively.

Solution of Moment Equations

Equations (16-20) are solved by approximating the distribution functions by a bimodal type of function similar to those in Ref. 25 but, instead of the spherically symmetric type used by Chung, they are made elliptical to account for the different effect of the buoyancy on the three components of the velocity. We let

$$f=f_1+f_2 \quad (21)$$

$$\theta=\theta_1+\theta_2 \quad (22)$$

$$n=n_1+n_2 \quad (23)$$

where for $V\geq 0$

$$f_1(y,U)=\frac{1}{\pi E_w(\pi E_v)^{1/2}}\exp\left\{-\frac{(u-u_1)^2+W^2}{E_w}-\frac{V^2}{E_v}\right\} \quad (24)$$

$$\theta_1=\theta_1(y,U), n_1=n_1(y,U), f_2=\theta_2=n_2=0 \quad (25)$$

and for $V<0$

$$f_2(y,U)=\frac{1}{\pi E_w(\pi E_v)^{1/2}}\exp\left\{-\frac{(u-u_2)^2+W^2}{E_w}-\frac{V^2}{E_v}\right\} \quad (26)$$

$$\theta_2=\theta_2(y,U), n_2=n_2(y,U), f_1=\theta_1=n_1=0 \quad (27)$$

In these distribution functions, the parameters $u_1, u_2, E_w, E_v, \theta_1, \theta_2, n_1$, and n_2 are determined by substituting Eqs. (21-27) into Eqs. (16-20) and solving the resulting simultaneous equations. The major difference between these distribution functions and those employed by Chung[25] lies in making E_v and E_w unequal, thus reflecting the different effects of the buoyancy force on the three components of the velocity.

Before discussing the solutions to the moment equations, some remarks on the large-scale relaxation rate β and the dissipative scale length λ are in order. By comparing the turbulent kinetic energy equation as obtained from Eq. (13) and that derived from the Navier-Stokes equation, Chung[23,24] found that the two are equivalent if

$$\beta=(3E)^{1/2}(2\Lambda) \quad (28)$$

It is assumed here that this relationship also holds when the buoyancy force is present. The effect of buoyancy will manifest through its influence on Λ and E. Further, from order of magnitude arguments, the ratio of the large scale to the dissipative scale is assumed to depend upon the turbulent Reynolds number in the following manner.[10] A similar relationship was also assumed by Donaldson[22]

$$(\Lambda/\lambda)^2=b\,\mathrm{Re}_\Lambda \quad (29)$$

with

$$\mathrm{Re}_\Lambda\equiv\Lambda(3E)^{1/2}/\nu \quad (30)$$

Substituting Eqs. (21-29) into Eqs. (16-20), one obtains after some algebraic rearrangement the following:

$$\alpha\psi = -\frac{\pi^{1/2}}{12}\left(\frac{L}{\Lambda}\right)\omega\left\{(1+6b)\phi^2-\alpha^2+(1+3b)\psi^2\right\} \tag{31}$$

$$\mathrm{Ri}\alpha f = \frac{\pi^{1/2}}{12}\left(\frac{L}{\Lambda}\right)\omega\left\{(2+6b)\alpha^2-2\phi^2-\tfrac{1}{2}\psi^2\right\} \tag{32}$$

$$0 = \frac{1}{12}\left(\frac{L}{\Lambda}\right)\omega\left\{(1+6b)\phi^2-\alpha^2-\tfrac{1}{2}\psi^2\right\} \tag{33}$$

$$2\alpha^2-\mathrm{Ri}f^2 = -\left[\frac{1+4b}{2\pi^{1/2}}\right]\left(\frac{L}{\Lambda}\right)\alpha f\omega \tag{34}$$

$$2\alpha^2-\mathrm{Ri}f\eta = -\left[\frac{1+4b}{2\pi^{1/2}}\right]\left(\frac{L}{\Lambda}\right)\alpha\eta\omega \tag{35}$$

with

$$\omega \equiv (4\phi^2+2\alpha^2+\psi^2)^{1/2} \tag{36}$$

and the parameters in the distribution functions have been made dimensionless according to

$$\psi \equiv \frac{u_1-u_2}{L(\mathrm{d}<u>/\mathrm{d}y)} \qquad \alpha \equiv \frac{E_v^{1/2}}{L(\mathrm{d}<u>/\mathrm{d}y)}$$

$$\phi \equiv \frac{E_w^{1/2}}{L(\mathrm{d}<u>/\mathrm{d}y)} \tag{37a}$$

$$f \equiv \frac{\theta_1-\theta_2}{L(\mathrm{d}<\theta>/\mathrm{d}y)} \qquad \eta \equiv \frac{n_1-n_2}{L(\mathrm{d}<n>/\mathrm{d}y)} \tag{37b}$$

where L is the characteristic length of the flow and Ri is the Richardson number defined by

$$\mathrm{Ri} \equiv \frac{(g/<\theta>)(\mathrm{d}<\theta>/\mathrm{d}y)}{(\mathrm{d}<u>/\mathrm{d}y)^2} \tag{38}$$

Physically, Ri is a measure of the work done by the fluid against the buoyancy force relative to the turbulent energy produced by shear. For $\mathrm{Ri}>0$, the flow is stable with respect to further generation of turbulence. Conversely, the flow is unstable for $\mathrm{Ri}<0$, and the buoyancy force aids in the generation of turbulence.

It is seen that the dimensionless functions are only dependent upon Ri, L/Λ, and b. Further, for any value of the Richardson number Eqs. (34) and (35) admit the solution

$$\eta = f \tag{39}$$

Referring to Eqs. (50) and (51), the statement of Eq. (39) means that the effect of buoyancy on the turbulent heat flux and diffusion flux is exactly the same. This particular result was also obtained previously by Donaldson.[21] Equations (31-35) can be rearranged to yield a closed-form solution for the quantities ψ/f, α/f and ϕ/f

$$\psi/f = \tfrac{1}{2}(G^2-4B)^{1/2}+\tfrac{1}{2}G \tag{40}$$

$$\alpha/f = -\{(\psi/12bf)[(\psi/f)-(1+6b)\mathrm{Ri}]\}^{1/2} \tag{41}$$

$$\phi/f = [2(\alpha/f)^2+(\psi/f)^2/2(1+6b)]^{1/2} \tag{42}$$

with

$$G = (1+6b)\mathrm{Ri}+2(1+4b)/\pi(1+2b) \tag{43}$$

$$B = \mathrm{Ri}\{[2(1+4b)(1+6b)/\pi(1+2b)]-6b\} \tag{44}$$

The solution is complete when Eq. (34) is used to express f in terms of the preceding functions.

$$\left(\frac{L}{\Lambda}\right)f = -\frac{4\pi^{1/2}}{1+4b}\left(\frac{1+6b}{3+6b}\right)^{1/2}\left\{1-\tfrac{1}{2}\mathrm{Ri}\left(\frac{f}{\alpha}\right)^2\right\}\left\{2+\frac{(\psi/f)^2}{(\alpha/f)^2}\right\}^{-1/2} \tag{45}$$

Substituting Eq. (45) into Eqs. (40-44) yields the solution for $(L/\Lambda)\alpha$, and $(L/\Lambda)\phi$. Finally, the turbulent kinetic energies, the turbulent fluxes, and the temperature fluctuation can be expressed in terms of these solutions as follows

$$<U^2>^* \equiv \frac{<U^2>}{\Lambda^2(\mathrm{d}<u>/\mathrm{d}y)^2} = \tfrac{1}{2}\left(\frac{L}{\Lambda}\phi\right)^2+\tfrac{1}{4}\left(\frac{L}{\Lambda}\psi\right)^2 \tag{46}$$

$$<V^2>^* \equiv \frac{<V^2>}{\Lambda^2(\mathrm{d}<u>/\mathrm{d}y)^2} = \tfrac{1}{2}\left(\frac{L}{\Lambda}\alpha\right)^2 \tag{47}$$

$$<W^2>^* \equiv \frac{<W^2>}{\Lambda^2(\mathrm{d}<u>/\mathrm{d}y)^2} = \tfrac{1}{2}\left(\frac{L}{\Lambda}\phi\right)^2 \tag{48}$$

$$<VU>^* \equiv \frac{<VU>}{\Lambda^2(\mathrm{d}<u>/\mathrm{d}y)^2} = \frac{1}{2\pi^{1/2}}\left(\frac{L}{\Lambda}\alpha\right)\left(\frac{L}{\Lambda}\psi\right) \tag{49}$$

$$<V\theta>^* \equiv \frac{<V\theta>}{\Lambda^2(\mathrm{d}<u>/\mathrm{d}y)(\mathrm{d}<\theta>/\mathrm{d}y)} = \frac{1}{2\pi^{1/2}}\left(\frac{L}{\Lambda}\alpha\right)\left(\frac{L}{\Lambda}f\right) \tag{50}$$

$$<Vn>^* \equiv \frac{<Vn>}{\Lambda^2(\mathrm{d}<u>/\mathrm{d}y)(\mathrm{d}<n>/\mathrm{d}y)} = \frac{1}{2\pi^{1/2}}\left(\frac{L}{\Lambda}\alpha\right)\left(\frac{L}{\Lambda}\eta\right) \tag{51}$$

$$<U\theta>^* \equiv \frac{<U\theta>}{\Lambda^2(\mathrm{d}<u>/\mathrm{d}y)(\mathrm{d}<\theta>/\mathrm{d}y)} = \tfrac{1}{4}\left(\frac{L}{\Lambda}\psi\right)\left(\frac{L}{\Lambda}f\right) \tag{52}$$

$$<(\theta-<\theta>)^2>^* \equiv \frac{<(\theta-<\theta>)^2>}{\Lambda^2(\mathrm{d}<\theta>/\mathrm{d}y)^2} = \tfrac{1}{4}\left(\frac{L}{\Lambda}f\right)^2 \tag{53}$$

The ratios of the eddy conductivity and diffusivity to the eddy viscosity can be obtained from Eqs. (49-51) as

$$K_H/K_M = f/\psi \tag{54}$$

$$K_S/K_M = \eta/\psi \tag{55}$$

where the eddy coefficients are defined as

$$<VU> = -K_M\frac{\mathrm{d}<u>}{\mathrm{d}y} \qquad <V\theta> = -K_H\frac{\mathrm{d}<\theta>}{\mathrm{d}y} \qquad <Vn> = -K_S\frac{\mathrm{d}<n>}{\mathrm{d}y} \tag{56}$$

Results and Discussions

The turbulent observable properties as expressed by the double correlations were computed for various values of the Richardson member by means of Eqs. (46-55). Note that, while the nondimensionalized double correlations are dependent only upon the Richardson number, the correlations themselves depend upon the large eddy scale and the gradients of the mean flow properties. Most experimental data do not report the gradients in sufficient detail to permit a direct comparison of the double correlations. Instead, only their ratios are compared in this paper whenever they are nondimensionalized by the same gradients. The most relevant experimental data for comparing with the present theory for a homologous shear flow were reported by Webster.[1] These data were measured in a wind tunnel of square cross section where linear velocity and temperature gradients in the vertical direction were produced by horizontal rods of varying diameter and varying heat input, equally spaced across the tunnel. Hot-wire anemometer data were collected at the center of the tunnel 1.73 m and 4.57 m from the shear grid.

Figures 1-4 show the comparison between Webster's data and the theoretical results from the present analysis for the ratios of total turbulent energy to its vertical component, the vertical to the streamwise turbulent energy, the spanwise to the streamwise turbulent energy, and the vertical heat flux to the streamwise heat flux, respectively. The constant b in Eq. (29) was chosen to be 0.15 so that at $Ri = 0$ the theoretical ratio of the total turbulent energy to its vertical component agrees with the experimental value of Webster in Fig. 1. However, the same value of b was employed throughout all the calculations. Considering the large scattering in the data, agreement with the experimental data is satisfactory except for the ratio of the vertical heat flux to the streamwise heat flux in Fig. 4. Here the theoretical results lie considerably above the experimental data, even though the tendency to decrease with increasing Richardson number is correctly predicted by the theory.

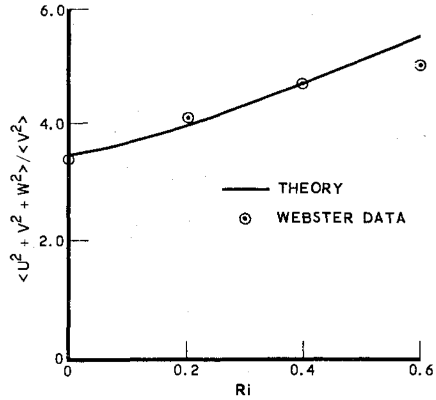

Fig. 1 Ratio of total to vertical turbulent kinetic energy.

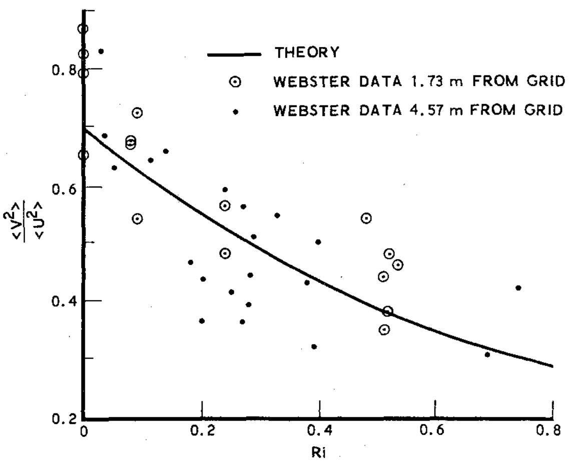

Fig. 2 Ratio of vertical to streamwise turbulent kinetic energy.

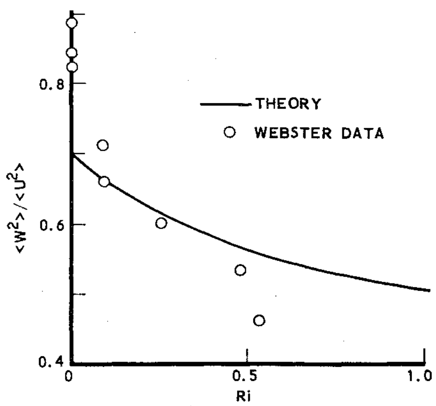

Fig. 3 Ratio of spanwise to streamwise turbulent kinetic energy.

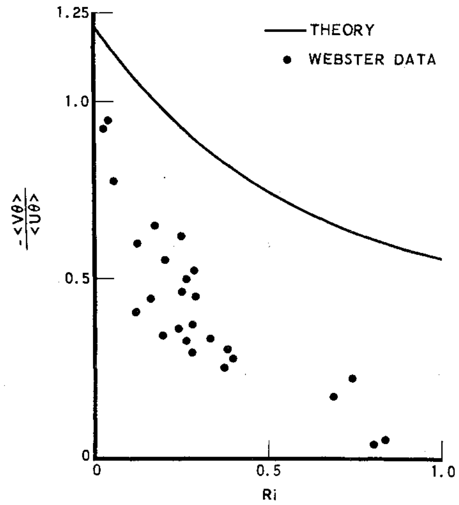

Fig. 4 Ratio of vertical to streamwise turbulent heat flux.

A reason for the discrepancy can be traced to the fact that the moment equations chosen to determine the parameters in the distribution function correspond to the governing equations for the three turbulent energy components and the vertical heat flux. It is, therefore, reasonable to expect better agreement with experimental data for these properties. The same cannot be said for the streamwise heat flux which was computed from the distribution function directly without solving its governing equation simultaneously. The discrepancy in the ratio of the two heat fluxes probably can be traced to the fact that the streamwise heat flux is not correctly predicted by the present theory.

Better agreement would be expected had Eqs. (10-12) been solved directly but, of course, with the accompanying mathematical complexity. The influence of Ri on the temperature fluctuation is shown in Fig. 5, where the zero subscript denotes the condition of $Ri = 0$. Implicit in this comparison with the data is the assumption that the large eddy scale does not change appreciably with Ri within the range covered by the experiment. The agreement between theory and data is satisfactory, even though the governing equation for the temperature fluctuation has not been included in the moment equations.

The ratio of eddy conductivity or diffusivity to the eddy viscosity is of considerable interest in studies of atmospheric pollution dispersion. As indicated previously, the present theory predicts the same behavior of the eddy conductivity and the eddy diffusivity toward stability conditions. Figure 6 compares Webster's data at the 4.57-m station with the present theory for K_H/K_M. At a limited range of the Richar-

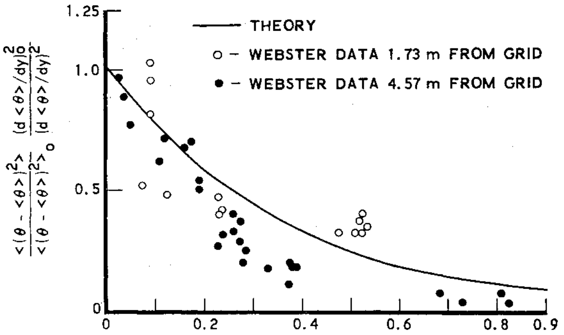

Fig. 5 Mean square temperature fluctuation vs Ri.

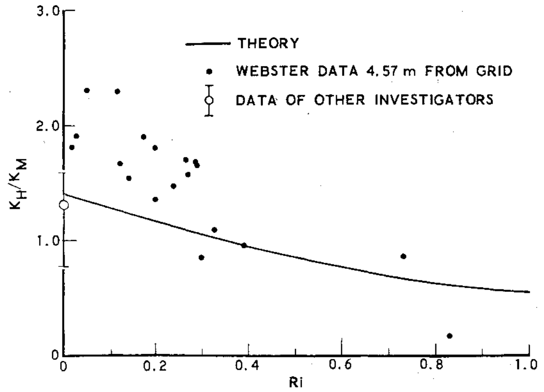

Fig. 6 Ratio of eddy conductivity to eddy viscosity.

dson number (Ri = 0.3 − 0.7), fair agreement is noted. However, at Richardson numbers near zero, the data lie considerably above the theoretical prediction. There is some evidence that Webster's data may be too high in this region. First, at Richardson numbers near zero the ratio of the eddy coefficients at the upstream station of 1.73 m measured as high as 6, which seems unrealistic. The data also shows that K_M increases rapidly while K_H remains relatively constant in the downstream direction, indicating that the steady-state condition was not attained for K_M in the experiment. Second, the present theory predicts $K_H/K_M = 1.39$ at Ri = 0 while a large number of independent experiments and theories, as listed in Table 1, reported a range of 0.8—1.6, with most of the data lying close to 1.3. However, Webster's data at the downstream station would extrapolate to a value of 2.2 at Ri = 0, which lies outside the range of past experimental data.

The most fundamental parameter for determining the stability condition of a stratified medium is the flux Richardson number R_f, defined as

$$R_f = g < V(\theta - <\theta>) > / <\theta> < VU > \mathrm{d} <u> / \mathrm{d}y \tag{57}$$

Table 1 Ratios of eddy conductivity and eddy diffusivity to eddy viscosity at the neutrally stable condition

K_H/K_M	K_S/K_M	Investigator	Reference
0.8		Swinbank	35
0.83—1.25		Johnson	36
1.0	1.0	Donaldson	21
1.3		Businger et al.	47
1.3		Page et al.	37
1.3		Reichardt	38
1.3		Senderikhina	46
	1.3	Rider	6
	1.33	Chung	25
1.1—1.4		Schlinger et al.	39
	1.3—1.4	Ellison and Turner	3
	1.4	Forstall and Shapiro	40
1.6	1.1	Pasquill	41
	1.6	Sherwood and Woertz	42

which is related to Ri by

$$R_f = (K_H/K_M)\,\mathrm{Ri} \tag{58}$$

A critical flux Richardson $R_{f,\mathrm{crit}}$ can be defined such that as $R_f \to R_{f,\mathrm{crit}}$, $K_H/K_M \to 0$, and as indicated by Monin and Yaglom,[48] Eq. (58) required that Ri → ∞. It can be easily shown from Eqs. (40-45) that all second-order correlations also vanish. Using Eqs. (40), (54), and (58), one obtains for Ri → ∞, $R_f \to 0.53$, $K_H/K_M \to 0$. Thus, there exists a critical flux Richardson number but no finite critical gradient Richardson number, as pointed out by Monin and Yaglom. From his experimental data Webster obtained a value of roughly 0.35 for $R_{f,\mathrm{crit}}$. Although the present theoretical critical flux Richardson number of 0.53 is higher than Webster's value, the data points he used for determining $R_{f,\mathrm{crit}}$ were highly inaccurate, corresponding to the two points on the extreme right of Fig. 6. In addition, two other data points which would have led to $R_{f,\mathrm{crit}}$ greater than 0.35 were discarded by Webster in the $R_{f,\mathrm{crit}}$ determination. Thus, the present theoretical value of $R_{f,\mathrm{crit}} = 0.53$ is at least consistent with Webster's data as a whole.

Besides the Webster data, both theoretical results and experimental data for the variation of K_H/K_M and K_S/K_M with Ri had previously been reported by a number of investigators over a wide range of Ri. All the experimental data, however, were obtained for flows quite different from the simple homologous type treated by the present theory. Figure 7 compares the present theoretical results with these data, which comprise two types. The data of Webster and of Merritt and Rudinger[2] were obtained in flows with temperature gradients, resulting in the measurement of K_H/K_M. All others were given as K_S/K_M as derived from measurements of local profiles of velocity and salinity. These include the data of Ellison and Turner,[3] Proudman's result based on the Kattegat data[43] and Jacobsen's result based on the Schultz Ground and the Raners Fjord data.[5] The theoretical curves were derived by Donaldson,[21] Munk and Anderson,[44] and Ellison,[45] who evaluated the values of K_{S_0}/K_{M_0} and the critical flux Richardson number from data in Ref. 3. The Munk and Anderson theoretical curve is dependent upon the value of K_{S_0}/K_{M_0}, which was set equal to 1.3 here, corresponding to the value most frequently reported. Referring to Fig. 7, one is immediately struck by the wide scattering of the data. The present theory appears to agree well with the theory of Munk and Anderson and lies much closer to the data of Webster, Jacobsen, and Proudman than the other data. On the other hand, the theoretical curves of Donaldson and Ellison appear to describe the data of Ellison and Turner and of Merritt and Rudinger better.

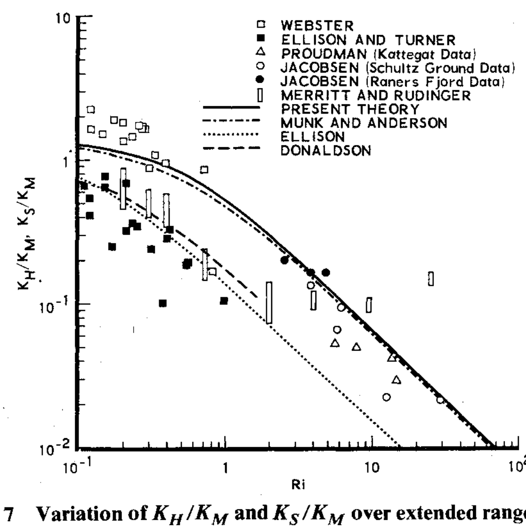

Fig. 7 Variation of K_H/K_M and K_S/K_M over extended range of Ri.

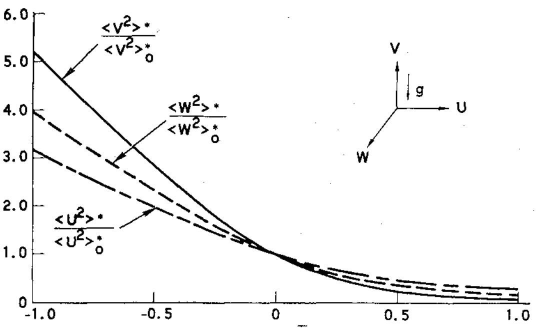

Fig. 8 Distribution of turbulent kinetic energy.

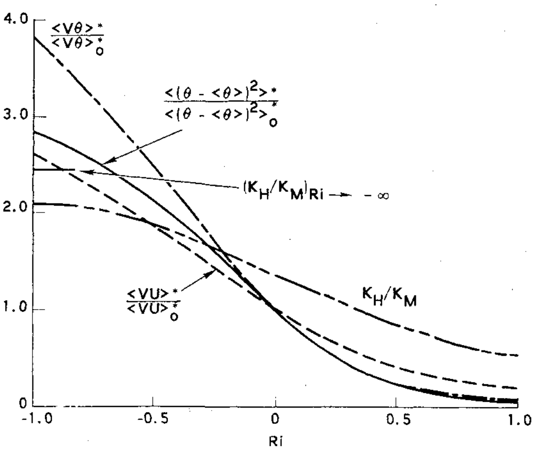

Fig. 9 Reynolds stress, heat flux, temperature fluctuation, and K_H/K_M vs Ri.

Additional theoretical results to demonstrate the effects of buoyancy on the turbulent field are presented in Figs. 8 and 9 for both unstable and stable conditions. All the important double correlations are enhanced at negative Ri and suppressed at positive Ri. Among the three components of the turbulent kinetic energy, the vertical turbulent motion is affected by the buoyancy to the greatest extent, as expected, since the buoyancy force has a direct influence on this motion. The turbulent motion in the other two directions is indirectly affected through the tendency of the turbulent field toward isotropy which is described mathematically by the last term in Eqs. (16-18). The streamwise turbulent kinetic energy is least affected by the buoyancy, since the interaction between the Reynold stress and the mean velocity gradient also influences the energy budget besides the tendency toward isotropy. Finally, the turbulent heat flux is more strongly influenced by the buoyancy than the Reynolds stress, resulting in an increase in K_H/K_M at unstable conditions and a decrease at stable conditions.

The comparison of the present theoretical results with experimental data herein is made on ratios of the turbulent properties in which the turbulent eddy scale does not enter explicitly so that the relative change of these turbulent properties with the Richardson number can be computed. To determine the absolute magnitude of the turbulent properties would require additional analysis of how the turbulent eddy scale changes with the Richardson number. While a useful relation among the large eddy scale, the characteristic length of the flow, and the velocity gradient was suggested by Chung[24] in the absence of buoyancy effects, extension of this relation to include buoyancy effects requires further study.

Conclusions

The effects of buoyancy on a turbulent shear flow was investigated through the application of Chung's statistical turbulent model to a homologous flow in the presence of the buoyancy force. The study reveals that all important double correlations are dependent not only upon the Richardson number but also upon the turbulent eddy scale and gradients of the mean flow properties. These double correlations are enhanced under unstable conditions and suppressed under stable conditions. The turbulent heat flux is more strongly affected by the buoyancy than the Reynolds stress, resulting in an increase of K_H/K_M under unstable conditions and a decrease under stable conditions. The effect of buoyancy on the turbulent heat transfer and mass transfer is found to be the same for a homologous flow under the condition of negligible chemical effects.

Comparison of the theoretical results with available experimental data leads to satisfactory agreement in the effect of buoyancy on the partition of the turbulent kinetic energy into its various components and on the mean square of the temperature fluctuation. The ratio K_H/K_M and K_S/K_M decreases at increasing Richardson numbers, as various experimental data show. The present theoretical values of K_H/K_M agree reasonably well with the theoretical curve of Munk and Anderson and with the data of Webster, Proudman and Jacobsen.

While the present study neglects the occurrence of chemical reactions and the associated heat effects on the temperature distribution and the buoyancy force, the formulation presented herein can readily be extended to turbulent-flow problems where chemical effects are relevant to the flow structure.

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